Enrollment No: ______________________ **Exam Seat No:**______________________

C. U. SHAH UNIVERSITY

Winter Examination - 2022

Subject Name : Linear Algebra

Subject Code : 5SC01LIA1 **Branch: M.Sc. (Mathematics)**

Semester : 1 **Date : 02/01/2023** **Time : 11:00 To 02:00** **Marks : 70**

<u>Instructions:</u>

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions.** **[07]**

a. Define : External Direct Sum. **(02)**

b. State First Homomorphism Theorem. **(02)**

c. Let V be a finite dimensional over F.If $T \in A(V)$ is right invertible then show that T is invertible. **(02)**

d. If $dim \dim V = 49$,then find $dim \dim A(V)$ and $dim \dim \hat{V}$. **(01)**

Q-2 **Attempt all questions** **[14]**

a. Let V be a finite dimensional vector space over F and W be subspace of V. Show that $\widehat{W}$ is isomorphic to $\frac{\hat{V}}{W^{\circ}}$ and **(08)**

$$dim \dim W^{\circ} = \dim \dim V - \dim \dim W .$$

b. If V is the internal direct sum of $U_1, U_2, \dots, U_n$ then show that V is isomorphic to the external direct sum of $U_1, U_2, \dots, U_n$. **(06)**

OR

Q-2

a. Let V and W be vector space over F of dimension m and n respectively. Then prove that $HOM(V, W)$ is of dimension mn over F. **(07)**

b. Let V be a finite dimensional vector space over F and W be subspace of V. Show that W is finite dimensional ,$dim \dim W \leq dim \dim V$ and **(07)**

$$dim \dim V/W = \dim \dim V - dim \quad \dim W .$$

Q-3 **Attempt all questions.** **[14]**

a. If A is an algebra over F with unit element then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. **(07)**

b. Let V be a finite dimensional vector space over F and $S,T \in A(V)$.show that **(07)**

i) $rank(ST) \leq rank(T)$

ii) $rank(TS) \leq rank(T)$

iii) If S is regular then $rank(ST) = rank(TS) = rank(T)$

OR

Q-3

a. Let V be a finite dimensional over F then prove that $T \in A(V)$ is invertible if and only if the constant term in minimal polynomial for T is nonzero. **(05)**

b. If V is finite dimensional over F, then prove that $T \in A(V)$ is regular if and only if T maps V on to V. **(05)**

c. Define $W^{\circ}$and show that if $U \subset W$ then $W^{\circ} \subset U^{\circ}$ where W and U are subspaces of vector space V. **(04)**

SECTION – II

Q-4 **Attempt the Following questions.** **[07]**

a Prove or disprove : $det\ det\ (A+B) = det\ det\ (A) + det\ det\ (B)$. **(02)**

b Prove that there do not exist $A, B \in M_n(F)$ such that $AB - BA = I$, where F is field with characteristic 0. **(02)**

c Find the symmetric matrix associated with the quadratic form $4x_1^2 + x_2^2 + x_3^2 + 6x_1x_2 - 10x_3 + 2x_2x_3$. **(02)**

d Define: Index of Nilpotence. **(01)**

Q-5 **Attempt all questions** **[14]**

a. Let V be a finite dimensional vector space over F and $T \in A(V)$. If all the characteristic roots of T are in F then prove that there is a basis of V with respect to which the matrix of T is upper triangular. **(08)**

b. Let V be a finite dimensional vector space over F and $T \in A(V)$. If all the characteristic roots of T are in F then show that T satisfies a polynomial of degree n over F. **(06)**

OR

Q-5

a. Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. If $V_1, V_2, \ldots, V_k$ are cyclic with respect to T such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ with $dim\ dim\ V_i = n_i (1 \leq i \leq k), n_1 \geq n_2 \geq \cdots \geq n_k$, and $U_1, U_2, \ldots, U_l$ are cyclic with respect to T such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_l$ with $dim\ dim\ U_i = m_i (1 \leq i \leq l)$, $m_1 \geq m_2 \geq \cdots \geq m_l$ then show that $k = l$ and $m_i = n_i$ for all i. **(07)**

b. Let V be a finite dimensional vector space over F and $T \in A(V)$. Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let $T_1 = T|_{V_1}$ and $T_2 = T|_{V_2}$. If the minimal polynomial of T_1 over F is $p_1(x)$ while minimal polynomial of T_2 over F is $p_2(x)$. Then show that minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$. **(07)**

Q-6 Attempt all questions **[14]**

a. Prove that the determinant of an upper triangular matrix is the product of its entries on the main diagonal. **(08)**

b. State and prove Cramer's rule. **(06)**

OR

Q-6

a. Let $A, B \in M_n(F)$,prove that $det\ det\ (AB) = det\ det\ A \cdot det\ det\ B$ **(07)**

b. Prove that determinant of a matrix and its transpose are same. **(04)**

c. Let F be a field. Then for $A, B \in M_n(F)$ and $\lambda \in F$, Prove: **(03)**

i) $tr(A + B) = tr(A) + tr(B)$

ii) $tr(AB) = tr(BA)$